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LETTER TO THE EDITOR

SU(2) parametrisation of Baxter's model

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Abstract. A parametrisation of the Boltzmann weights of Baxter's symmetric eight-vertex model is derived, using Cherednik's representation of the Zamolodchikov algebra, in terms of theta functions based on the weight lattice of SU(2). The Yang–Baxter equations are satisfied as a consequence of the associativity of the Zamolodchikov algebra. The derivation makes the connection of Baxter's model to the spin-$\frac{1}{2}$ representation of SU(2) explicit.

Baxter's symmetric eight-vertex model [1] is central to studies of exactly solvable lattice models in two dimensions: all known off-critical solutions can be related to it, either as special cases, or generalisations of it. For reviews of recent developments, see [2, 3].

The basic observation that underlies generalisations of Baxter's model can be phrased as follows: the model is related to the spin-$\frac{1}{2}$, or two-dimensional representation of the Lie group SU(2), the group of angular momentum, in the sense that the variable placed on each bond can take either one of two states; these correspond to the weights of the two-dimensional representation of SU(2): spin-up and spin-down. The purpose of this work is to make this connection explicit.

Accordingly, we can think of generalisations that correspond to larger irreducible representations of SU(2), and beyond that to any irreducible representation of any Lie group [4–10].

A basic step towards solving a lattice model exactly, is to obtain a parametrisation of the Boltzmann weights that satisfies the Yang–Baxter equations. If the model is off-critical, then the parametrisation is given in terms of elliptic functions: infinite series in a complex parameter q, the 'nome', that parametrises the departure from criticality. In the critical limit, q tends to zero, and the Boltzmann weights reduce to trigonometric functions.

However, Baxter's original parametrisation of the Boltzmann weights does not make the connection with SU(2) manifest. In this work, we wish to derive a parametrisation of Baxter's model, in a way that makes this connection clear. The point of the excercise is to formulate things in such a way that generalisation to models based on other representations and/or other Lie groups becomes straightforward.

The derivation is based on a representation of the Zamolodchikov [11] algebra proposed by Cherednik [12]. In addition to being simple and systematic, it seems extendible to more general vertex models, where the structure of the Boltzmann weights is not transparent, or even unknown.

A necessary condition for a lattice model to be exactly solvable is that the number of conserved charges to be equal to the number of degrees of freedom. In the infinite-lattice limit, one should have an infinite number of conserved charges. The

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operators corresponding to these charges can be obtained as terms in a Taylor expansion of the transfer matrix with respect to a 'spectral parameter'.

The dependence of the transfer matrix on the spectral parameter is equivalent to the presence of a one-parameter family of commuting transfer matrices. A sufficient condition that ensures the presence of a family of commuting transfer matrices is that the Boltzmann weights of the model satisfy the Yang–Baxter equations. Obtaining new exactly solvable models starts with finding new solutions to the Yang–Baxter (YB) equations.

In [11], Zamolodchikov proposed that the YB equations can be regarded as the associativity condition of a non-commutative algebra: consider an algebra generated by the set $\{A_i(x)\}$, where i is a discrete index, x is a continuous parameter, and the group relation is the 'braiding' operation:

$$A_i(x) * A_j(y) = S_{ij}^{kl}(x-y) A_k(y) A_l(x). \tag{1}$$

If the algebra is associative:

$$(A_i(x) * A_j(y)) * A_k(z) = A_i(x) * (A_j(y) * A_k(z)) \tag{2}$$

then the structure constants $S_{ij}^{kl}(x)$ satisfy

$$S_{\alpha\alpha'}^{\gamma\gamma'}(u) S_{\gamma\alpha''}^{\beta\gamma''}(u+v) S_{\gamma'\gamma''}^{\beta'\beta''}(v) = S_{\alpha'\alpha''}^{\gamma'\gamma''}(v) S_{\alpha\gamma''}^{\gamma\beta''}(v+u) S_{\gamma\gamma'}^{\beta\beta'}(u) \tag{3}$$

which are the YB equations, once we interpret $S_{ij}^{kl}(u)$ as a Boltzmann weight, and the argument u as a spectral parameter. We can find new solutions to the YB equations by looking for new realisations of the Zamolodchikov algebra (1).

Note that one can, in principle, think of the YB equations as functional equations, and proceed to find solutions to them. But this is not easy. Our impression is that it should be easier to find new candidates for associative Zamolodchikov algebras.

In [12] Cherednik proposed a realisation of the Zamoldochikov algebras, in terms of a ring of theta functions, that leads directly to off-critical Boltzmann weights that satisfy the YB equations. Let us begin with a few definitions.

Our main reference to theta functions is [13]. The classical theta functions, of degree m, nome q, characteristic μ, and complex argument z are defined as

$$\Theta_{\mu,m,t}(z) = \mathrm{e}^{-2\pi i m t} \sum_{\gamma \in Z + \nu/m} q^{m(\gamma)^2} \mathrm{e}^{-2\pi i m \gamma . z}. \tag{4}$$

Notice that this definition is different from that used in [20, 21]: we have only one characteristic rather than two, since the second can always be absorbed in z. Furthermore, the exponential factor, with parameter t, is added so that the properties of the theta functions under modular transformations (reparametrisations that cannot be deformed to the identity) can be written in a simple way. We will not need these here, so t will always be set to zero in this work. We will drop the third subscript of Θ from now on.

For a given degree m, there are m independent theta functions, with characteristics $\mu = \{0, \ldots, m-1\}$, in the sense that they span the space of theta functions of degree m: $\widetilde{Th}_m$. This statement is made explicit in (16) below.

The direct sum of spaces of degree-m theta functions, $\widetilde{Th} = \oplus_{m \in Z} \widetilde{Th}_m$, $\widetilde{Th}$ is called a ring of theta functions. For precise definitions and details see [13].

Next we turn to Cherednik's realisation of the Zamoldochikov algebra [12]. Since Cherednik's work is quite technical, we will follow the clear exposition given in appendix II of [14], with modifications that suit our purposes.

Think of $\widetilde{Th}_m$ as a vector space spanned by the independent theta functions, and define a 'state' in $\widetilde{Th}_m$ by $|\Theta_{\mu,m}(z)\rangle$.

Consider the operators that act on $|\Theta_{\mu,m}(z)\rangle$ as follows:

$$A_{i,m}(x)|\Theta_{j,m}(y)\rangle = \Theta_{i,m}(y-x)|\Theta_{j,m}(y-\eta)\rangle. \tag{5}$$

That is, they act on $|\Theta_{\mu,m}(z)\rangle$ by shifting the argument by a constant η, and multiplication by a degree-m theta function. The results of multiplying two theta functions of degrees m and n, can be expanded in terms of degree $m+n$ theta functions:

$$\Theta_{\mu_1,m_1}(z_1)\Theta_{\mu_2,m_2}(z_2) = \sum_{\gamma \in Z \bmod (m_1+m_2)Z} c_\gamma \Theta_{\mu_1+\mu_2+m_1\gamma, m_1+m_2}(z_1+z_2) \tag{6}$$

where the expansion coefficients are

$$c_\gamma = \Theta_{m_2\mu_1 - m_1\mu_2 + m_1 m_2 \gamma, m_1 m_2 (m_1+m_2)}(z_1 - z_2). \tag{7}$$

Thus the operators $A_{i,m}(z)$ take us from one subspace in the ring to another. Clearly, the action of $A_{i,m}(z)$, as given in (5), is not the most general that one can consider, but it is precisely what we need in this work.

Next, we consider the quadratic actions:

$$\begin{aligned} &A_{i,m}(x)A_{j,m}(y)|\Theta_{j,m}(z)\rangle \\ &A_{k,m}(y)A_{l,m}(x)|\Theta_{j,m}(z)\rangle. \end{aligned} \tag{8}$$

Both operations take the initial state $|\Theta_{j,m}(y)\rangle$ from $\widetilde{Th}_m$ to some final state in $\widetilde{Th}_{3m}$. Regarding the final states as vectors in $\widetilde{Th}_{3m}$, we can relate them by a transformation matrix:

$$A_{i,m}(x)A_{j,m}(x)|\Theta_{j,m}(y)\rangle = S_{ij}^{kl}(x-y)A_{k,m}(y)A_{l,m}(x)|\Theta_{j,m}(y)\rangle. \tag{9}$$

Given (6), it is natural to express the operators $A_{i,m}(z)$ in terms of theta functions:

$$A_{i,m}(z) \to \Theta_{i,m}(z). \tag{10}$$

Using (10) in (9), we obtain

$$\Theta_{i,m}(x)\Theta_{j,m}(y) = S_{ij}^{kl}(x-y)\Theta_{k,m}(y)\Theta_{l,m}(x). \tag{11}$$

Since the algebra of products of theta functions is associative, the structure constants, $S_{ij}^{kl}(x-y)$, satisfy the YB equations. They are given in terms of theta functions and, therefore, are candidates for the Boltzmann weights of an off-critical model. This is, essentially, Cherednik's realisation of the Zamolodchikov algebra. Notice that it requires that the monomials of degree two in the theta functions, used to represent the generators $\{A_{i,m}(x)\}$, be independent.

But how can we obtain the Boltzmann weights of a specific lattice model? Let us consider Baxter's symmetric eight-vertex model. The allowed vertices are shown in figure 1.

The model is symmetric since we take

$$\omega_1 = \omega_2 = a \qquad \omega_3 = \omega_4 = b \qquad \omega_5 = \omega_6 = c \qquad \omega_7 = \omega_8 = d. \tag{12}$$

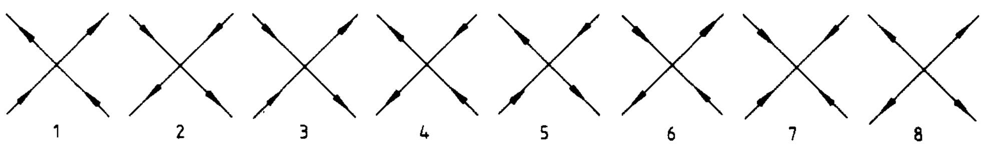

Figure 1. The vertices of the symmetric eight-vertex model.

The general eight-vertex model, with no symmetry as in (12), has not been solved. Let us consider a general vertex with bonds labelled as in figure 2: We can think of a vertex as a diagram describing an elastic two-body scattering, with two incoming, and two outgoing states. Notice that we could choose any two states to be incoming, and the other two outgoing. Given the symmetries of the model, there are three independent choices, corresponding to the three channels of elastic two-body scattering. We will take the states i and j as incoming, and k and l as outgoing.

We wish to associate a theta function with each state with a definite index, then compute the Boltzmann weight from (11). Since each index has two values, we wish to have a two-dimensional vector space of theta-functions. For that we take $m=2$ in (4). The rest of the computation is straightforward, and has been outlined in [14].

We wish to repeat this computation in a way that makes the connection with the spin-$\frac{1}{2}$ representation of SU(2) explicit. For that we propose to use a Cherednik-type representation of the Zamolodchikov algebra based on theta functions related to SU(2). We will see that they reduce to those used above.

There is a direct generalisation of the classical theta functions (4) to functions defined on regular r-dimensional lattices [15]. Following the notation of [13], these are defined as

$$\Theta^{L}_{\mu,m}(z)=\sum_{\gamma\in L+\mu/m} q^{m(\gamma|\gamma)}\,\mathrm{e}^{-2\pi i m(\gamma|z)} \tag{13}$$

where L is an r-dimensional lattice, μ and z are r-dimensional vectors.

An important class of theta functions are those based on lattices associated with Lie groups, as will be briefly explained in the following subsection. The case we are interested in is the simplest non-trivial one: SU(2).

Our main references on lattices associated with Lie groups are [16, 17]. For reviews see [18, 19]. Here we wish to recall some basic facts. In a matrix representation of a rank r Lie group, r generators can be simultaneously diagonalised. They form the 'Cartan subalgebra' of the group. For SU(2), $r=1$, and only one generator can be diagonalised. The states that form the irreducible representations of the group are eigenvectors of the Cartan subalgebra. The corresponding eigenvalues are the 'weights' associated with the representation.

For a rank r group, the weights form r-dimensional vectors, considered as arrows with their tail-ends at the origin; their end points are the vertices of an r-dimensional lattice called the weight lattice of the group. For SU(2) the lattice is one-dimensional, and shown in figure 3:

The normalisation of the weights will be explained below. The weights of the adjoint representation—for SU(2) this is the spin-1, or three-dimensional representation—are called the 'roots'. The set of all roots are linear combinations of r 'simple' roots, that generate a 'root lattice'. The root lattice of SU(2) is shown in figure 4. The normalisation

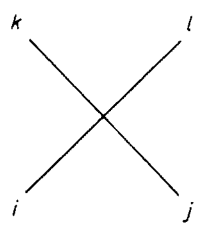

Figure 2. A general vertex.

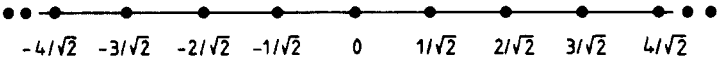

Figure 3. The weight lattice of SU(2).

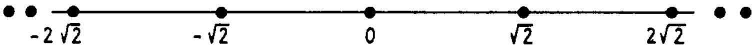

Figure 4. The root lattice of SU(2).

of the roots is such that the norm of a simple root is 2. The weight lattice is dual to the root lattice: it is generated by the 'fundamental weights', which are defined as the duals to the simple roots. This explained the normalisation of the weights in figure 3. Notice that the root lattice is a sublattice of the weight lattice. This is the case for all Lie groups, with the exception of E_8, where the root lattice is self-dual and identical to the weight lattice.

The inner product of the vectors that generates a lattice define a quadratic form that acts as a metric on the lattice. The quadratic form corresponding to a root lattice is the 'Cartan matrix'. The quadratic form of the weight lattice is the inverse Cartan matrix. In the case of SU(2), these are scalars.

We can write down a theta function based on a lattice L as in [13]. We are interested in the case where the characteristics take values in the dual lattice L^*. In that case, the vector space Th_m is spanned by a set of theta functions of degree m, with characteristics $\mu \in L \bmod mL^*$ [13]. The obvious choice is to write down theta functions based on the root lattice L of a group with characteristics taking values in the weight lattice L^* modulo the roots.

We can do that for the lattices based on SU(2), and obtain a parametrisation of the symmetric eight-vertex model that way, since the unit cell of the SU(2) root lattice contains precisely two vertices from the weight lattice. However, the characteristics will not take values directly in the weights of the spin-$\frac{1}{2}$ representation. Therefore, we wish to proceed differently.

We start with theta functions based on L^*, the *weight* lattice of SU(2), with characteristics taking values in the root lattice L. But since L is only a sub-lattice of L^*, we will have to work with level $m > 1$ theta functions, so that the characteristics take values in $L \bmod mL^*$, and choose m such that there are precisely two independent characteristics. This way, the characteristics, and consequently the incoming and outgoing states in a vertex, can be directly related to the weights of the spin-$\frac{1}{2}$ representation.

It is convenient to work in terms of an orthonormal basis, and to encode the geometry of the lattice in terms of a matrix that acts as a metric on the lattice. In the case of the root lattices, this is the Cartan matrix.

In the normalisation where the norm of the simple roots is 2, the Cartan matrix of SU(2) is the scalar 2. The inverse Cartan matrix is $\frac{1}{2}$. The degree m theta functions based on the weight lattice of SU(2) are:

$$\Theta^{L}_{\mu,m,t}(z) = \mathrm{e}^{-2\pi i m t} \sum_{\gamma \in Z+\mu/m} q^{m\gamma.1/2.\gamma} \mathrm{e}^{-2\pi i m \gamma.1/2.z} \qquad \mu \in Z_m \tag{14}$$

where $Z_m \equiv \{0, 1, \ldots, m-1\}$. The only possibility that leads to a two-dimensional space of theta functions, where the characteristics have the correct periodicity properties is

$m=4$. In this case, the characteristics, that take values in $L \bmod 4L^*$, can be chosen as $\{-\frac{1}{2}, \frac{1}{2}\}$ or $\{0, 1\}$. The two choices are equivalent, since they describe different bases of the same vector space. The set $\{-\frac{1}{2}, \frac{1}{2}\}$ coincides with the weights of the spin-$\frac{1}{2}$ representation.

To simplify the computations, we will make use of the invariance under change of basis, take the characteristics in the set $\{0, 1\}$, and write our theta functions as

$$\Theta^{L}_{\mu,4}(z) = \sum_{\gamma \in Z+\mu/2} q^{2\gamma^2} e^{-4\pi i \gamma . z} \qquad \mu \in \{0, 1\}. \tag{15}$$

Note that these are identical to what one obtains starting with theta functions based on the root lattice and $m=1$. These are precisely the theta functions used in [14] to compute the weights of the symmetric eight-vertex model. Here, we had a clear look at where they come from. Next we proceed with the derivation of the Boltzmann weights.

For each vertex, we associate theta functions with the incoming states, take their product, and expand it in terms of a basis of degree $2m$ theta functions using (6). Then we do the same for the outgoing states, and relate the two expansions using (11). Next, we solve for the Boltzmann weights, using the orthogonality relation

$$\int \Theta^{L}_{\mu,m}(z)\overline{\Theta^{L}_{\mu',m}(z')}\, dn \sim \delta_{L+\mu, L+\mu'} \tag{16}$$

where $z' = z + \text{constant}$, and the precise definition of the integration measure dn is given in [13]. The proportionality factor will not be interesting to us, since it cancels out in the final results. The answer is

$$\begin{aligned}
a &= S^{00}_{00}(z) = S^{11}_{11}(z) = \frac{[\Theta_0(-z+\eta)\Theta_0(z+\eta) - \Theta_2(-z+\eta)\Theta_2(z+\eta)]}{[\Theta_0^2(-z+\eta) - \Theta_2^2(-z+\eta)]} \\
b &= S^{01}_{10}(z) = S^{10}_{01}(z) = \frac{[\Theta_1(-z+\eta)\Theta_3(z+\eta) - \Theta_1(z+\eta)\Theta_3(-z+\eta)]}{(\Theta_1^2(-z+\eta) - \Theta_3^2(-z+\eta))} \\
c &= S^{10}_{10}(z) = S^{01}_{01}(z) = \frac{[\Theta_1(-z+\eta)\Theta_1(z+\eta) - \Theta_3(-z+\eta)\Theta_3(z+\eta)]}{(\Theta_1^2(-z+\eta) - \Theta_3^2(-z+\eta))} \\
d &= S^{11}_{00}(z) = S^{00}_{11}(z) = \frac{[\Theta_0(-z+\eta)\Theta_2(z+\eta) - \Theta_0(z+\eta)\Theta_2(-z+\eta)]}{[\Theta_0^2(-z+\eta) - \Theta_2^2(-z+\eta)]}
\end{aligned} \tag{17}$$

I have checked that (17) satisfies the YB equations using series expansions up to ten orders in q, for different values of the arguments. Equation (17) is not manifestly identical to Baxter's parametrisation, which is:

$$\begin{aligned}
a &= \Theta_{0,1,0}(2\eta)\Theta_{0,1,0}(\lambda-\eta)\Theta_{0,1,1}(\lambda+\eta) \\
b &= \Theta_{0,1,0}(2\eta)\Theta_{0,1,1}(\lambda-\eta)\Theta_{0,1,0}(\lambda+\eta) \\
c &= \Theta_{0,1,1}(2\eta)\Theta_{0,1,0}(\lambda-\eta)\Theta_{0,1,0}(\lambda+\eta) \\
d &= \Theta_{0,1,1}(2\eta)\Theta_{0,1,1}(\lambda-\eta)\Theta_{0,1,1}(\lambda+\eta).
\end{aligned} \tag{18}$$

An analytic proof that both parametrisations are equivalent is beyond the scope of this work. Here we content ourselves by remarking that in the scaling limit, both parametrisations coincide, up to overall factors, and for suitable choices of the constants η. Furthermore, as we have seen, the theta functions used in this work reduce to those obtained in [14], and their results, obtained following the same method as ours, were shown to be equivalent to those of Baxter.

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